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THE THERMAL BENDING OF COMPOSITE PLATES

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Abstract—Within the framework of classical plate theory, it is shown in this paper that if we know the deflection and moments when a subregion of an isotropic homogeneous infinite plate is heated to a prescribed temperature moment, then the corresponding results for a two-phase infinite plate are directly deducible by differentiation of the known results when the interface is a straight line or a circle. Full appreciation of this fact can pave the way for the avoidance of a mathematically sophisticated process of solution. The analysis is based upon the Green's function method and covers the case of a concentrated couple applied near the interface of two bonded semi-infinite elastic plates, as well as that of a circular inclusion in a field of uniform twist.

INTRODUCTION

IN THE first part of this paper, we are concerned with the bending problem of two different, isotropic classically elastic semi-infinite plates which are bonded along the line $y = 0$. A subregion of the plate occupying the region $y > 0$ is heated to a prescribed temperature moment. Outside of that region, the temperature remains unchanged. Using real function approach, in conjunction with the Green's function method, it is shown that there exist simple linear differential relations between components of the field in the composite plate and those of the corresponding field in a homogeneous infinite plate and that these relations do not depend upon the shape of the hot region. Full appreciation of this fact can result in a marked economy of effort in the actual construction of the desired results. It can also make possible a simple organisation of the calculations. Although the established differential relations are mathematically similar to those obtained by Aderogba and Berry [1] in the corresponding plane stress problem, the fields of the present problem depend upon *all* the elastic constants. However, the dependence is again by way of only two combinations of the material parameters.

The second part of the paper considers the case of an isotropic infinite plate containing a circular inhomogeneity, a subregion of the matrix being heated to a prescribed temperature moment. Following the procedure employed in the first part of the paper, it is again shown that the fields in the infinite and composite plate are functionally dependent. This conclusion is again analogous to that reached by Aderogba [2, 3] in the corresponding plane stress problem, and holds good when the inclusion is in a field of uniform twist. An extension to the case of a concentrated couple applied near the interface of two bonded semi-infinite plates is next discussed.

Let us remark that all these results are mathematically analogous to the sphere theorems of Weiss [4], Butler [5] and Cholton [6], which are various versions of the three-dimensional counterpart of the hydrodynamic circle theorem of Milne-Thomson [7].

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Finally, reference must be made to the classical papers by Mindlin and Cheng [8] and Dundurs and Guell [9, 10], whose original contributions inspired the present investigation. In [8] Mindlin and Cheng established a functional relationship between the elastic fields in an infinite and a semi-infinite solid, under the assumption that the two solids contain an identical thermal inclusion. The generalization of this three-dimensional result to the case of bonded semi-infinite solids was thoroughly considered in [10].

BASIC EQUATIONS

In an orthogonal co-ordinate system (x, y), the Cartesian form of the field equations associated with the thermal bending of thin isotropic plates is [11].

$$\begin{aligned} M_1 &= -D\left[\nu\nabla^2 W+(1-\nu)\frac{\partial^2 W}{\partial x^2}+(1+\nu)\varepsilon T\right],\\ M_2 &= -D\left[\nu\nabla^2 W+(1-\nu)\frac{\partial^2 W}{\partial y^2}+(1+\nu)\varepsilon T\right],\\ M_{12} &= -D(1-\nu)\frac{\partial^2 W}{\partial x\,\partial y}; \end{aligned} \tag{1}$$

$$\begin{aligned} V_1 &= -D\frac{\partial}{\partial x}\left[(2-\nu)\nabla^2 W-(1-\nu)\frac{\partial^2 W}{\partial x^2}\right],\\ V_2 &= -D\frac{\partial}{\partial y}\left[(2-\nu)\nabla^3 W-(1-\nu)\frac{\partial^2 W}{\partial y^2}\right], \end{aligned} \tag{2}$$

$$\nabla^2\nabla^2 W+(1+\nu)\varepsilon\nabla^2 T = 0. \tag{3}$$

In equations (1–3), T stands for the temperature moment, W for the deflection, V_1 and V_2 for the associated Kirchhoff shears, M_1 and M_2 for the bending moments, and M_{12} for the twisting moment. The elastic constants ε, ν and D respectively stand for the co-efficient of linear thermal expansion, Poisson's ratio and flexural rigidity of the plate; while the Laplacian operator ∇^2 is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}. \tag{4}$$

The foregoing field equations are generally accompanied by appropriate boundary, continuity and regularity conditions.

Once the temperature moment T is known, a particular solution of (3) is derivable on the basis of the theory of the potential and is as follows:

$$4\pi W(x, y) = -(1+\nu)\varepsilon\iint_S T(\xi, \eta)\log[(x-\xi)^2+(y-\eta)^2]\,d\xi\,d\eta, \tag{5}$$

where S is the domain of definition of T. The results we seek are motivated by the striking similarity between the particular solution (5) and the displacement potential of Goodier [12] for two-dimensional thermal inclusion problems.

BONDED SEMI-INFINITE PLATES

The problem of two bonded semi-infinite dissimilar plates is considered here. Material 1 occupies $y > 0$, material 2 $y < 0$, and an arbitrary region S (Fig. 1) of material 1 is heated to a prescribed temperature moment T which vanishes outside this region. Assuming that there is no fault or crack along the bond, we are required to satisfy the equilibrium equation (3) subject to the following continuity conditions [13] on $y = 0$:

$$W^{(1)} = W^{(2)}, \qquad \frac{\partial W^{(1)}}{\partial y} = \frac{\partial W^{(2)}}{\partial y}, \qquad\qquad (6)$$
$$M_2^{(1)} = M_2^{(2)}, \qquad V_2^{(1)} = V_2^{(2)},$$

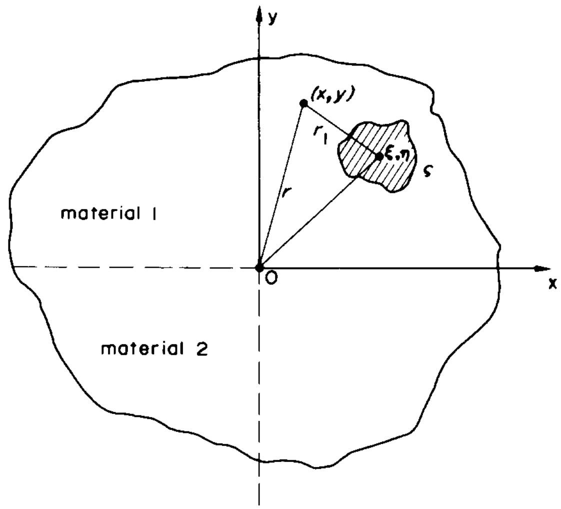

FIG. 1.

where superscripts in brackets differentiate the two materials. It will be shown that the following theorem holds:

Theorem 1

If a subregion of an isotropic homogeneous infinite plate is heated to a prescribed temperature moment which produces the deflection $W(x, y)$ and moments $M_1(x, y)$, $M_2(x, y)$, $M_{12}(x, y)$ whose singularities are all in the half-plane $y > 0$, then on introducing a different material into the region $y < 0$ the new deflection and moments become, for $y > 0$,

$$W^{(1)} = W(x, y) - \alpha\left[1 - 2y\frac{\partial}{\partial y}\right]W(x, -y),$$
$$M_{12}^{(1)} = M_{12}(x, y) - \alpha\left[1 + 2y\frac{\partial}{\partial y}\right]M_{12}(x, -y),$$
$$M_1^{(1)} = M_1(x, y) + \left[1 + 2\alpha + g(\alpha - 1) + 2\alpha y\frac{\partial}{\partial y}\right]M_1(x, -y), \qquad\qquad (7)$$
$$M_2^{(1)} = M_2(x, y) - \left[1 + g(\alpha - 1) - 2\alpha y\frac{\partial}{\partial y}\right]M_2(x, -y);$$

and for $y < 0$,

$$
\begin{aligned}
W^{(2)} &= (1-\alpha)W(x, y),\\
M_{12}^{(2)} &= g(1-\alpha)M_{12}(x, y),\\
M_1^{(2)} &= g(1-\alpha)M_1(x, y),\\
M_2^{(2)} &= g(1-\alpha)M_2(x, y),
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
\alpha &= \frac{\kappa^{(1)}(g-1)}{4+\kappa^{(1)}(g-1)},\\
g &= \frac{D^{(2)}\kappa^{(2)}}{D^{(1)}\kappa^{(1)}},\\
\kappa^{(i)} &= 1-\nu^{(i)}, \quad i = 1, 2.
\end{aligned}
\tag{9}
$$

Proof

First, we must establish equations (7) and (8) for the Green's function, that is, for the case of an infinitesimal bending hot spot of unit intensity located in material 1 at the point $x = \xi$, $y = \eta$ (Fig. 1).

To this end, we recall that in the absence of material discontinuities, the deflection produced in the homogeneous infinite plate by such a bending hot spot is [11]†

$$w^o = \sigma \log r_1, \tag{10}$$

where

$$\sigma = -\frac{(1+\nu)\varepsilon}{2\pi} \tag{11}$$

while

$$r_{1,2}^2 = (x-\xi)^2+(y\mp\eta)^2. \tag{12}$$

The associated moments m_1^o, m_2^o and m_{12}^o are obtainable by formal substitution of (10) into (1), with $T = 0$.

Guided by (10), we now construct the additional expression for the deflection in material 1 as follows:

$$
\begin{aligned}
W = &-A_1\sigma \log r_2\\
&+A_2\sigma y\frac{\partial}{\partial y}(\log r_2)
\end{aligned}
\tag{13}
$$

while the complete deflection in material 2 is taken in the following form:

$$
\begin{aligned}
W = &-A_3\sigma \log r_1\\
&-A_4\sigma y\frac{\partial}{\partial y}(\log r_1).
\end{aligned}
\tag{14}
$$

† See p. 58.

The constants A_i, $i = 1, 2, 3, 4$, are to be determined from the continuity conditions (6), using formulae (1) and (2) for the moments and Kirchhoff shears. The satisfaction of these conditions yields a system of four linear equations whose solution is

$$\begin{aligned} A_1 &= \alpha, & A_2 &= 2\alpha, \\ A_3 &= \alpha - 1, & A_4 &= 0, \end{aligned} \tag{15}$$

where α is once again given by (9). The formal solution of the auxiliary bending hot spot problem is now complete. The additional field produced in $y > 0$ is obtained by substituting (13) into (1) and (2) while the total field produced in $y < 0$ is obtained by a similar substitution of (14) into (1) and (2). A complete explicit listing of the final results for the deflection and moments induced in the composite plate is as follows:

$$\begin{aligned} \frac{w^{(1)}}{\sigma} &= \log r_1 - \alpha \log r_2 + 2\alpha y(y+\eta)/r_2^2 \\ \frac{m_1^{(1)}}{\sigma D^{(1)}} &= -\kappa^{(1)}[(y-\eta)^2 - (x-\xi)^2]/r_1^4 \\ &\quad + \alpha(4 - 3\kappa^{(1)})[(y+\eta)^2 - (x-\xi)^2]/r_2^4 \\ &\quad + 4\alpha\kappa^{(1)} y(y+\eta)[(y+\eta)^2 - 3(x-\xi)^2]/r_2^6 \\ \frac{m_2^{(1)}}{\sigma D^{(1)}} &= -\kappa^{(1)}[(y-\eta)^2 - (x-\xi)^2]/r_1^4 \\ &\quad + \alpha(4 - \kappa^{(1)})[(y+\eta)^2 - (x-\xi)^2]/r_2^4 \\ &\quad - 4\alpha\kappa^{(1)} y(y+\eta)[(y+\eta)^2 - 3(x-\xi)^2]/r_2^6 \\ \frac{m_{12}^{(1)}}{\sigma D^{(1)}} &= 2\kappa^{(1)}(x-\xi)[(y-\eta)/r_1^4 + \alpha(y+\eta)/r_2^4] \\ &\quad - 4\alpha\kappa^{(1)} y(x-\xi)[3(y+\eta)^2 - (x-\xi)^2]/r_2^6; \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{w^{(2)}}{\sigma} &= (1-\alpha)\log r_1 \\ \frac{m_1^{(2)}}{\sigma D^{(1)}} &= -g(1-\alpha)\kappa^{(1)}[(y-\eta)^2 - (x-\xi)^2]/r_1^4; \\ m_2^{(2)} &= -m_1^{(2)}, \\ \frac{m_{12}^{(2)}}{\sigma D^{(1)}} &= 2g(1-\alpha)\kappa^{(1)}(y-\eta)(x-\xi)/r_1^4. \end{aligned} \tag{17}$$

It is not difficult to verify that equations (16)–(17) satisfy the field equations, as well as the continuity conditions (6).

If we turn now to a comparison of the composite plate solution with the infinite plate solution (obtainable from (16) and (17) by merely proceeding to the limit as $g \to 1$), we readily conclude that these solutions are linked by differential relations of precisely the same form as equations (7) and (8), that is,

$$w^{(1)} = w^o(x, y) - \Box w^o(x, -y), \tag{18}$$

$$w^{(2)} = (1-\alpha)w^o(x, y), \text{ etc.} \tag{19}$$

where

$$\Box = \alpha\left[1 - 2y\frac{\partial}{\partial y}\right]. \tag{20}$$

Suppose now that the prescribed temperature moment T is a function of x and y. Then the elastic fields induced in the composite plate can be obtained by multiplying every term of equations (16) and (17) by $T(\xi, \eta)$ and then integrating with respect to ξ and η over the region S of the definition of T. For example,

$$\begin{aligned} W^{(1)} &= \iint_S T(\xi, \eta) w^{(1)} \, \mathrm{d}\xi \, \mathrm{d}\eta \\ &= \iint_S T(\xi, \eta)[w^o(x, y) - \Box w^o(x, -y)] \, \mathrm{d}\xi \, \mathrm{d}\eta, \text{ from (19)}, \\ &= \iint_S T(\xi, \eta) w^o(x, y) \, \mathrm{d}\xi \, \mathrm{d}\eta - \Box \iint_S T(\xi, \eta) w^o(x, -y) \, \mathrm{d}\xi \, \mathrm{d}\eta \\ &= W(x, y) - \Box W(x, -y). \end{aligned} \tag{21}$$

The other formulae in (7) and (8) can be similarly established.

Therefore, if we have previously determined the deflection and moments in an isotropic homogeneous infinite plate subjected to a local temperature moment, the corresponding results for two bonded semi-infinite plates are computable by differentiation. We note that only the first partial derivatives of the homogeneous infinite plate deflection and moments are required in the actual construction of the composite plate solution, and hence the general relations (7) and (8) avoid the use of comparatively unwieldy formulae. It may also be observed that, as far as the interface $y = 0$ is concerned, the elastic components in the composite plate are constant multiples of the corresponding results in the homogeneous infinite plate, which might prove attractive to those who are primarily interested in numerical results. Finally, from the definition of α in (9), it is seen that the elastic components in the composite plate depend upon the Poisson's ratio of material 2. This is in strong contrast to the stress state in bonded half-planes subjected to plane hot spots [1].

INFINITE PLATE WITH A CIRCULAR INCLUSION

In the present section, we consider the case in which an isotropic homogeneous infinite plate (material 1) contains a central circular inclusion (material 2) of radius a. A certain region S (Fig. 2) of material 1 is heated to a prescribed temperature moment T which vanishes outside S. The deflection and moments in the composite plate are to be determined and expressed in terms of the corresponding results in the homogeneous infinite plate.

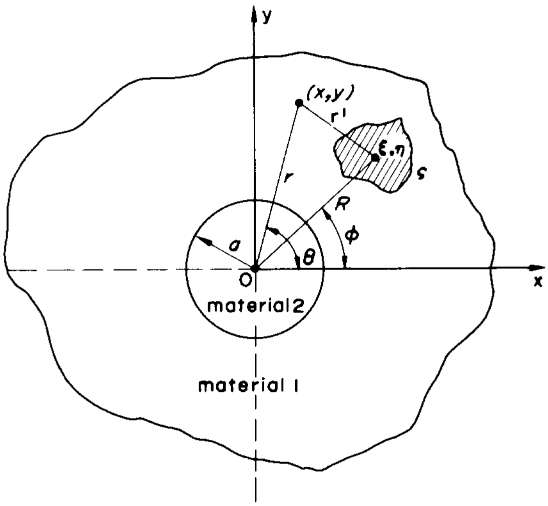

FIG. 2.

To achieve this goal, we introduce the polar co-ordinates (r, θ) which are connected with the Cartesian co-ordinates (x, y) through $x = r\cos\theta$, $y = r\sin\theta$. Then, according to the classical theory of the bending of thin plates, equations (1) and (2) transform into:

$$
\begin{aligned}
M_r &= -D\left[\nu\nabla^2 W + (1-\nu)\frac{\partial^2 W}{\partial r^2} + (1+\nu)\varepsilon T\right],\\
M_\theta &= -D\left[\nabla^2 W - (1-\nu)\frac{\partial^2 W}{\partial r^2} + (1+\nu)\varepsilon T\right],\\
M_{r\theta} &= -D(1-\nu)\left[\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r} - \frac{\partial}{\partial r}\right)\right]W,\\
V_r &= -\left[D\frac{\partial}{\partial r}(\nabla^2 W) + \frac{1}{r}\frac{\partial}{\partial\theta}(M_{r\theta})\right],\\
V_\theta &= -\left[D\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2 W) + \frac{\partial}{\partial r}(M_{r\theta})\right],
\end{aligned}
\tag{22}
$$

where the Laplacian operator now takes the form

$$
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{23}
$$

To equations (3), (22) and (23) we adjoin the continuity conditions

$$
\begin{aligned}
W^{(1)} &= W^{(2)}, \qquad \frac{\partial W^{(1)}}{\partial r} = \frac{\partial W^{(2)}}{\partial r},\\
M_r^{(1)} &= M_r^{(2)}, \qquad V_r^{(1)} = V_r^{(2)}
\end{aligned}
\tag{24}
$$

which must hold when $r = a$ for $0 \leq \theta < 2\pi$, where superscripts in brackets again relate to materials 1 and 2. Also, since there is no loading at infinity, at least M_r and V_r must tend to zero there.

It will be shown that the following theorem holds:

Theorem 2

If a subregion of an isotropic homogeneous infinite plate is subjected to a prescribed temperature moment, characterized by the deflection $W(r, \theta)$ and moments $M_r(r, \theta)$, $M_\theta(r, \theta)$, $M_{r\theta}(r, \theta)$ all the singularities of which occur in the region $r > a$, then on introducing a different elastic material into the region $r < a$, the new deflection and moments are given by

$$
\begin{aligned}
W^{(1)} &= W(r, \theta) - \alpha\left[1 - (r/a^2)(r^2 - a^2)\frac{\partial}{\partial r}\right] W(\rho, \theta), \\
M_r^{(1)} &= M_r(r, \theta) + \alpha(a/r)^2[(4/\kappa^{(1)}) - (a/r)^2] M_r(\rho, \theta) \\
&\quad - \alpha(a^2/r^4)(r^2 - a^2)\left(4 - r\frac{\partial}{\partial r}\right) M_r(\rho, \theta) \\
M_\theta^{(1)} &= M_\theta(r, \theta) + \alpha(a/r)^2[4 - (4/\kappa^{(1)}) - (a/r)^2] M_\theta(\rho, \theta) \\
&\quad - \alpha(a^2/r^4)(r^2 - a^2)\left(4 - r\frac{\partial}{\partial r}\right) M_\theta(\rho, \theta) \\
M_{r\theta}^{(1)} &= M_{r\theta}(r, \theta) + \alpha(a/r)^2[(a/r)^2 - 2] M_{r\theta}(\rho, \theta) \\
&\quad + \alpha(a^2/r^4)(r^2 - a^2)\left(4 - r\frac{\partial}{\partial r}\right) M_{r\theta}(\rho, \theta);
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
W^{(2)} &= (1 - \alpha) W(r, \theta), \\
M_r^{(2)} &= g(1 - \alpha) M_r(r, \theta), \\
M_r^{(2)} &= g(1 - \alpha) M_\theta(r, \theta), \\
M_{r\theta}^{(2)} &= g(1 - \alpha) M_{r\theta}(r, \theta),
\end{aligned}
\tag{26}
$$

where α and g are once again given by (9), while

$$\rho = a^2/r. \tag{27}$$

Proof

As in the proof of Theorem 1, the first step is to establish (25) and (26) for the problem of a bending hot spot of unit intensity located at the point $x = \xi = R\cos\phi, y = \eta = R\sin\phi$ (Fig. 2). To this end, we set $w^{(1)} = w^o + w_1$ for the total deflection in material 1 and write $w^{(2)}$ for the total deflection in material 2, where w^o is once again given by (10). Since no further singularities are required in the composite plate, the deflection produced in the homogeneous infinite plate suggests that the residual fields w_1 and $w^{(2)}$ must be defined as follows:

$$w_1 = -\beta_1\sigma \log r_3 + \beta_2\sigma r[(r/a)^2-1]\frac{\partial}{\partial r}(wgr_3), \tag{28}$$

$$w^2 = -\beta_3\sigma \log r_1 + \beta_4\sigma r[(r/a)^2-1]\frac{\partial}{\partial r}(\log r_1) \tag{29}$$

where β_1, β_2, β_3 and β_4 are four as yet undetermined constants, while

$$r_3^2 = \rho^2 + R^2 - 2\rho R\cos(\theta-\phi). \tag{30}$$

It is easily verified that equations (28) and (29) satisfy the equilibrium equation (3), with $T = 0$. Expressions for the required fields may now be formally obtained by substitution of (28) and (29) into (22). When these results are used in conjunction with (24), we obtain the following simple expressions for the constants:

$$\begin{aligned} \beta_1 &= \alpha, & \beta_2 &= \alpha, \\ \beta_3 &= \alpha - 1, & \beta_4 &= 0, \end{aligned} \tag{31}$$

where α is given by (9).

From (28)–(31), it is now readily confirmed that the complete deflection outside the circle $r = a$ is

$$w^{(1)} = \sigma[\log r_1 - \alpha\log r_3] - \sigma\alpha\rho[(r/a)^2-1][\rho - R\cos(\theta-\phi)]/r_3^2, \tag{32}$$

while the complete deflection inside the circle $r = a$ is

$$w^{(2)} = \sigma(1-\alpha)\log r_1. \tag{33}$$

Comparison of (32) with the infinite plate deflection (10) shows that these two deflections are connected through

$$w^{(1)} = w^o(r,\theta) - \Delta w^o(\rho,\theta), \tag{34}$$

where

$$\Delta = \alpha\left\{1 - r[(r/a)^2-1]\frac{\partial}{\partial r}\right\}. \tag{35}$$

We are now in a position to calculate the deflection at any point $x = r\cos\theta$, $y = r\sin\theta$ due to an arbitrary distribution of temperature moment T over the region S containing the point $\xi = R\cos\phi$, $\eta = R\sin\phi$. Indeed, by the Green's function method,

$$\begin{aligned} W^{(1)} &= \iint_S RT(R,\phi)w^{(1)}\,\mathrm{d}R\,\mathrm{d}\phi \\ &= \iint_S RT(R,\phi)w^o(r,\theta) - \Delta w^o(\rho,\phi)\mathrm{d}R\;\mathrm{d}\phi, \\ &= \iint_S RT(R,\phi)w^o(r,\theta) - \Delta\iint_S RT(R,\phi)w^o(\rho,\phi)\,\mathrm{d}R\,\mathrm{d}\phi \\ &= W(r,\theta) - \Delta W(\rho,\theta), \end{aligned} \tag{36}$$

which agrees with the first differential relation in (25). The other formulae in (25) and (26) can be similarly established.

The same method can be employed in treating the case when the temperature moment is prescribed *within* the inhomogeneity, the conclusion being that the disturbed and undisturbed fields are linked by differential relations of precisely the same form as equations (25) and (26). Of course, in this case, the superscript 1 refers to the inhomogeneity while the superscript 2 refers to the surrounding medium.

SOME POSSIBLE EXTENSIONS AND EXAMPLES

1. *Composite plate under a concentrated couple*

With a view to generalizing Theorem 1 to other loading conditions, we consider here the problem of a single concentrated couple M applied as shown in Fig. 3 near the interface of two bonded semi-infinite plates at the point $x = \xi$, $y = \eta$. The results for the action of a concentrated couple are easily generalized further to other loading conditions.

In the absence of material discontinuities, we know that the deflection produced in an isotropic homogeneous infinite plate by one concentrated couple is biharmonic, being given by [15]†

$$W = e(y-\eta)\log(r_1), \tag{37}$$

where

$$e = -M/4\pi D^{(1)}$$

while r_1 is once again given by (12). Notice that, apart from a multiplicative constant, equation (37) contains a term due to a bending hot spot located at (ξ, η) in an homogeneous infinite plate. Consequently, some of the earlier results are applicable here.

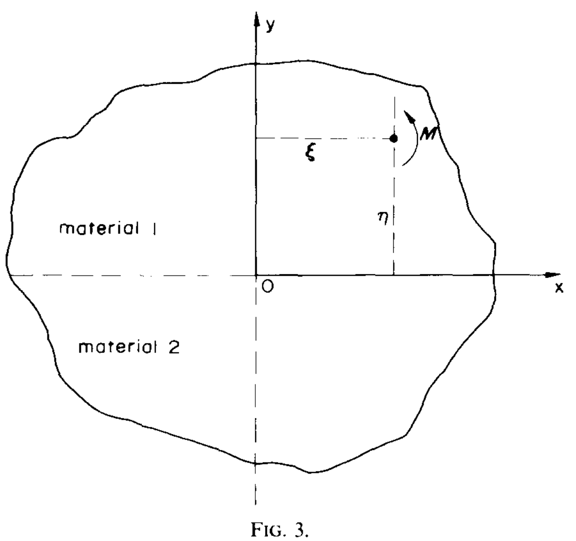

FIG. 3.

† See p. 326.

We shall not go through the process of solution of the residual problem, in order to save space and to avoid repetition. We therefore proceed directly to the final results and conclusion, namely the representation of the deflections in the composite plate in terms of the harmonic function

$$\psi(x, y) = \log(r_1) \tag{38}$$

in the homogeneous infinite plate deflection (37):

$$\begin{aligned} W^{(1)} &= e(y-\eta)\psi(x, y)+e\alpha\eta\left[1-2y\frac{\partial}{\partial y}\right]\psi(x, -y)+e\left[\beta\int\psi(x, -y)\,\mathrm{d}y-\alpha y\psi(x, -y)\right], \\ W^{(2)} &= -e\left[\beta\int\psi(x, y)\,\mathrm{d}y+y(\alpha-2\beta-1)\psi(x, y)\right]-e\eta(1-\alpha)\psi(x, y), \end{aligned} \tag{39}$$

where α is once again given by (9), while

$$\begin{aligned} \beta &= \frac{2[f(1-\alpha)-(1+\alpha)]}{[f(\kappa^{(2)}-4)-\kappa^{(1)}]}, \\ f &= D^{(2)}/D^{(1)}. \end{aligned} \tag{40}$$

The validity of equations (39) can be verified by formally substituting these results into (1) and (2) and thence into (6), bearing in mind the fact that the integrals in these results are harmonic.

The theoretical significance of equations (7) and (39) is that if the deflection in an isotropic homogeneous infinite plate is representable in the form

$$W^o = \psi_1(x, y)+(y-\eta)\psi_2(x, y), \tag{41}$$

where ψ_1 and ψ_2 are harmonic functions, then, irrespective of the nature of the loading condition, the deflections produced in two bonded isotropic semi-infinite plates occupying the regions $y > 0$ and $y < 0$ are expressible in terms of the functions ψ_1 and ψ_2 as follows:

$$\begin{aligned} W^{(1)} = {} & \psi_1(x, y)+(y-\eta)\psi_2(x, y)-\alpha\left[1-2y\frac{\partial}{\partial y}\right][\psi_1(x, -y)-\eta\psi_2(x, -y)] \\ & -\alpha y\psi_2(x, -y)+\beta\int\psi_2(x, -y)\,\mathrm{d}y, \end{aligned} \tag{42}$$

$$W^{(2)} = (1-\alpha)[\psi_1(x, y)-\eta\psi_2(x, y)]-(\alpha-2\beta-1)y\psi_2(x, y)-\beta\int\psi_2(x, y)\,\mathrm{d}y, \tag{43}$$

it being understood that the loading is applied to the material occupying the region $y > 0$.

2. *Circular inclusion in a field of uniform twist*

Here we consider the case in which an infinite plate containing a circular inhomogeneity of radius a is subjected to a uniform twist at infinity, defined by the moments $M_1 = M_2 = 0$, $M_{12} = M_o$ which produce the deflection

$$W = Nr^2 \sin 2\theta, \qquad N = -M_o/2\kappa^{(1)}D^{(1)} \tag{44}$$

in the homogeneous infinite plate.

The derivation of the solution of the residual problem follows precisely the same procedure as in the proof of Theorem 2. In the interest of brevity, we quote only the final results:

$$\begin{aligned} W^{(1)} &= N[r^2-\alpha(a/r)^2(2r^2-a^2)]\sin 2\theta, \\ M_r^{(1)} &= [1+\alpha(a/r)^2\{(4/\kappa^{(1)})-4+3(a/r)^2\}]\sin 2\theta, \\ M_\theta^{(1)} &= -[1-\alpha(a/r)^2\{(4/\kappa^{(1)})-3(a/r)^2\}]\sin 2\theta, \\ M_{r\theta}^{(1)} &= -[1+\alpha(a/r)^2\{2-3(a/r)^2\}]\cos 2\theta, \end{aligned} \tag{45}$$

$$\begin{aligned} W^{(2)} &= N(1-\alpha)r^2\sin 2\theta, \\ M_r^{(2)} &= -M_\theta^{(2)} = g(1-\alpha)\sin 2\theta, \\ M_{r\theta}^{(2)} &= -g(1-\alpha)\cos 2\theta, \end{aligned} \tag{46}$$

where the superscript 2 again refers to the inclusion while the superscript 1 refers to the surrounding medium.

By comparing the composite plate solution thus assembled with the infinite plate solution

$$(W, M_r, M_\theta) = (Nr^2, 1, -1)\sin 2\theta, \qquad M_{r\theta} = -\cos 2\theta, \tag{47}$$

it is easily confirmed that these solutions are connected according to equations (25) and (26). This conclusion suggests that Theorem 2 is applicable to some isothermal problems and may therefore be modified to read as follows:

Let a circular disc of radius a be perfectly bonded to an infinite plate. Let the matrix be subjected to a loading condition which in the homogeneous infinite plate produces a deflection which is the gradient of a harmonic function† whose singularities are all outside the disc. Then the deflection and moments produced in the composite plate are directly deducible from the corresponding fields in the homogeneous infinite plate by means of equations (25 and 26).

The proof of this modified theorem is analogous to that of Theorem 1 of [3] and will not be given here.

3. *Rectangular hot area in a composite plate*

We finally consider the moments produced in two bonded semi-infinite plates at a temperature moment zero except for a rectangular region within which the temperature moment is T_o and uniform. The dissimilar semi-infinite plates occupy the regions $y > 0$ and $y < 0$ and the hot region is the domain S ($|x| < a$, $|y-c| < b$), where $c \geq b$. Along $y = 0$, we require the satisfaction of the continuity conditions (6), where, as usual, the superscripts 1 and 2 relate to $y > 0$ and $y < 0$, respectively.

The solution of the foregoing problem for the homogeneous infinite plate follows from the formal substitution of (5) into (1) and (2), whilst the solution of the residual problem can be derived by merely invoking the relations (7) and (8). It is easily shown that the final results are expressible in terms of logarithms and inverse tangents.

† Except possibly within the loaded region.

Of particular interest are the special values of the results along the straight line ($y = 0$) of material discontinuity:

$$
\begin{aligned}
M_1^{(1)} &= m_o[2(1+\alpha)+g(\alpha-1)]\psi_o \\
M_2^{(i)} &= m_o g(\alpha-1)\psi_o, \; i = 1, 2, \\
M_{12}^{(1)} &= m_o(1-\alpha)\log R_1, \\
M_1^{(2)} &= m_o g(1-\alpha)\psi_o, \\
M_{12}^{(2)} &= m_o g(1-\alpha)\log R_1,
\end{aligned}
\tag{48}
$$

where

$$
\begin{aligned}
\psi_o &= \arctan[(c+b)/(x-a)]+\arctan[(c-b)/(x+a)] \\
&\quad -\arctan[(c+b)/(x+a)]-\arctan[(c-b)/(x-a)], \\
R_1^2 &= \frac{(x-a)^2+(c+b)^2}{(x-a)^2+(c-b)^2}\cdot\frac{(x+a)^2+(c+b)^2}{(x+a)^2+(c-b)^2} \\
m_o &= -D^{(1)}(1-\nu^{(1)})(1+\nu^{(1)})\varepsilon^{(1)}T_o/2\pi.
\end{aligned}
\tag{49}
$$

These results are comparable with the explicit expressions contained in [16] for the corresponding components in the analogous plane stress problem.

CONCLUSION

Our main objective in this paper has been to show that when an arbitrary temperature moment is prescribed in one of two bonded semi-infinite plates, or in an infinite plate containing a circular inclusion, it is possible to reduce the calculation of the deflection and moments in the composite plate to a process of differentiation if the corresponding results in the homogeneous infinite plate are available. Consideration of the method of solution shows that the established differential relations are valid for any temperature moment prescribed over regions of any shape. It is hoped that the results presented here will enable elasticians to foresee the results of other analogous, and certainly more useful problems.

The extension to the case of the bending of bonded orthotropic semi-infinite plates under a general transverse loading condition has been completed and will be discussed in another paper, the structure of the resultant relations being essentially similar to those contained in [17] which was devoted to the corresponding plane stress problem.

Perhaps it will prove possible to generalize Theorem 2 to anisotropic elasticity. It would also be interesting to prove that the basic relations (7) and (8) are valid for the case of an isotropic composite plate strip simply supported along the edges and subjected to any local temperature moment in the interior of one of the bonded plates.

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Абстракт—В рамках классической теории пластинок, доказывается в работе, что для известных прогибов и моментов, когда подобласть изотропной, однородной, бесконечной пластинки нагрета к заданной температуре, тогда можно непосредственно вычитать соответствующие результаты для двух-фазной бесконечной пластинки путем дифференцирования известных результатов для границы раздела в виде прямой линии или круга. Полное определение этого факта может прокладывать путь в целью избежания математически усложненного процесса решения. Анализ основан на методе функции Грина и охватывает случай концентрической пары сил, приложенной близи границы раздела двух ограниченных, полубесконечных пластинок, а также случай круглого включения в полю однородного кручения.